Physics 111

Lecture 19

Thursday, November 4, 2002

- Ch 13: Springs
  - Force, Work, Potential Energy
  - Period
  - Amplitude
  - Functions: x, v, a
  - Pendulum Motion

Announcements

This week’s lab will be on oscillations. There will be a short quiz.

Chapter 13

Oscillations

We’ve already studied some vibrational motion, when we examine the curious behavior of springs and objects that interact with them. We will expand our studies to objects that behave similarly to our spring, such as the pendulum and rotating objects.

Spring Review!

\[ F_{\text{spring}} = -k\Delta x \]

- Force exerted by the spring in compression/stretching by an amount \( \Delta x \).

\[ W = -\frac{1}{2}k(\Delta x)^2 \]

- Work done by the spring in compression/stretching by an amount \( \Delta x \).

\[ U_i = \frac{1}{2}kx^2 \]

- Potential Energy of a spring is

Concept Quiz!

Worksheet #1

CAUTION
WATCH YOUR STEP

Oscillations

Our socialite, bored from counting Krugerrands, decides to play with his new spring toy.

Worksheet #2

Predict the motion of the mass oscillating on this spring as explicitly as possible.

Assume no friction and no air resistance.
Our socialite, bored from counting Krugerrands, decides to play with his new spring toy.

Now, sketch a plot of the height of the block above the floor as a function of time.

What kind of mathematical functions (with which you’re familiar) result in such a pattern?

Worksheet #2

An object in simple harmonic motion displays an acceleration that is proportional to the displacement and in the opposite direction.

Not simple harmonica music!

An object in simple harmonic motion can be described by trigonometric functions.

These trigonometric functions complete one cycle (peak-to-peak) over what angular displacement?

That is to say, what is $\theta_1 - \theta_0$?

$2\pi$

So what if we observe our spring to oscillate with a period of 3 seconds. How would we write our function for the height of our block versus time?

$h = A \cos\left(\frac{2\pi}{T} t\right)$
What is the Period of this wave? 3 seconds

What is the Frequency of this wave? 1/3 s⁻¹

\[ h = A \cos \left( \frac{2 \pi t}{T} \right) = A \cos (2 \pi f t) \]

Where have we seen this quantity before? 2\( \pi f \)

Angular Frequency!!!

Circular motion!

Now...you've got to be thinking to yourself...

Well...I'm gonna tell ya!

What does a spring have to do with circular motion?

Here's my tennis ball on a string again.

What if I were now to shine a spotlight on this system from the right side and look at the motion of the shadow on a wall to the left?

We're looking down on the plane of motion.

Q: What will a graph of the height of the shadow on the wall look like?

A: Just like the position vs time for the mass on a spring!

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
So, the trigonometric functions describe BOTH circular motion (Ch 6, 10) and the motion of a mass on a spring (Ch 6)!

On what quantities did the period of our tennis ball moving in a circle at the end of a string depend?

- its speed
- the radius of the circle.

\[ T = \frac{2\pi r}{v} \]

If we look at the motion of the shadow, what will be the amplitude of the oscillation?

So we might surmise that the period for our mass-spring system will also be related to the amplitude of the motion!

\[ T = \frac{2\pi A}{v_{\text{max}}} \] where \( A \) represents the amplitude of the oscillation.

Our wealthy socialite has been playing with his spring in a gravitational field...(that complicates our problem--although it leads to the same answer).

Let's instead look for the moment at a mass-spring system that's oriented horizontally on a frictionless surface--so we can take gravity out of the picture.

Let's use energy conservation to explain the motion.

If we measure \( x \) as the distance from the equilibrium (natural length) of the spring, what will the energy of the system be when the mass reaches its maximum displacement to the right at \( x = A \)?

\[ K = 0 \quad U_x = \frac{1}{2} k A^2 \quad E_{\text{total}} = \frac{1}{2} k A^2 \]

And what will the energy of the system be when the mass passes through its equilibrium position at \( x = 0 \)?

\[ K = \frac{1}{2} m v_{\text{max}}^2 \quad U_x = 0 \quad E_{\text{total}} = \frac{1}{2} m v_{\text{max}}^2 \]
Conservation of energy requires

\[ E_{\text{total}} = \text{constant} \]

which implies that

\[ \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 \quad \Rightarrow \quad \frac{A^2}{v_{\text{max}}} = \frac{m}{k} \]

Let's use energy conservation to explain the motion.

We've Done It!

Recall that the period is given by

\[ T = \frac{2\pi A}{v_{\text{max}}} \]

Ah ha!

Now we see that the period depends ONLY upon two constant quantities: the mass and the spring constant!

\[ A \text{ and } v_{\text{max}} \text{ are dependent on one another!} \]

Let's look at one more characteristic of our mass-spring system:

How does the velocity of the mass relate to its displacement from the equilibrium position?

\[ E_{\text{total}} = \text{constant} = \frac{1}{2} k A^2 \]

Using the point of maximum potential energy...

The total energy at any other point in the motion of the block is a mixture of kinetic and potential given by:

\[ E_{\text{total}} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \]

So

\[ \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 + \frac{1}{2} k x^2 \]
Solving for \( v \), we find

\[
\frac{1}{2} m v^2 = \frac{1}{2} k A^2 - \frac{1}{2} k x^2
\]

Solving for \( v \), we find

\[
v = \frac{k}{m} (A^2 - x^2)
\]

Note the two solns

\[
v = \pm \sqrt{\frac{k}{m} (A^2 - x^2)}
\]

Let's use energy conservation to explain the motion.

\[
\frac{1}{2} m v^2 = \frac{1}{2} k A^2 - \frac{1}{2} k x^2
\]

\[
v = \pm \sqrt{\frac{k}{m} (A^2 - x^2)}
\]

\[
\frac{1}{2} m a = -k x
\]

This equation has a solution given by...

\[
x = A \cos(\omega t + \phi)
\]

\[
x = A \sin(\omega t + \phi)
\]

...where \( A \) is the amplitude and \( \phi \) is the “phase.”

Notice that the maximum velocity, (which occurs when the sine function takes a value of 1) is

\[
v = -v_{\text{max}} \sin(\omega t + \phi)
\]

And the maximum acceleration, (which occurs when the cosine function takes a value of 1) is

\[
a = -a_{\text{max}} \cos(\omega t + \phi)
\]
The motion of a bob at the end of the pendulum can be thought of as acting under a restoring force, just like the mass on a spring. But this time, the agent responsible for the restoring force is gravity.

Here, the component of the gravitational force that acts perpendicularly to the string supporting our yellow ball results in the motion of the pendulum.

Using geometry, we find the restoring force is

\[ F_s = -mg \sin \theta \]

As long as the amplitude of the oscillations remains relatively small, we can use the small angle approximation for \( \sin \theta \) and get

\[ F = -mg \theta \]

\[ F = -mg \frac{s}{L} \]

which now has the form \( F = -kx \)

Now using our results from the motion of the mass-spring system to describe the motion of the pendulum with the substitution

\[ k \rightarrow -\frac{mg}{L} \]

and we find the period of our pendulum to be

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

Good for SMALL amplitude oscillations!