Physics 111

Lecture 17

Thursday, October 28, 2004

- Ch 8: Work done by force at an angle
  Power
- Ch 10: Rotational Kinematics
  angular displacement
  angular velocity
  angular acceleration

Announcements

- Wednesday, 8 - 9 pm in NSC 118/119
- Sunday, 6:30 - 8 pm in CCLIR 468

This week's lab will be a workshop of physics. Bring your ranking tasks book. No quiz.

Concept Quiz!

Worksheet #1

Ch 8: Conservation of Energy

Labs Meet This Week

This week’s lab will be a workshop of physics. Bring your ranking tasks book. No quiz.

Ch 7: Work and Kinetic Energy

Let’s now look at a force that is applied at an angle to the direction of motion.

How does our problem change?

Skip to result

Frictionless pond of ice

Ch 7: Work and Kinetic Energy

If the block remains on the ice, and the man keeps the angle between the rope and the ice constant, and the man exerts a constant force, the work done on the block is....

Frictionless pond of ice
We look at the component of the force that is in the direction of motion of the object (i.e., along the direction of \( \mathbf{s} \)).

\[ T \cos \theta \]

\( \theta \) is the angle between the force \( T \) and the direction \( \mathbf{s} \).

**Frictionless pond of ice**

**Work!**

\[ W = F \Delta s \]

Notice: Work is a **scalar quantity**, which means that you **do NOT specify a direction** associated with work. Work only has a magnitude.

Let's introduce a new mathematical operation to express the type of product we need to calculate in order to correctly compute work.

**Scalar Product** (a.k.a. Dot Product)

The scalar (or dot) product of any two vectors \( \vec{A} \) and \( \vec{B} \)

\[ \vec{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \]

and

\[ \vec{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \]

is given by the expression

\[ \vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z \]

Which mathematically says:

- multiply the \( x \)-components of the two vectors;
- add the result to the product of the \( y \)-components of the two vectors; and finally
- add the result to the product of the \( z \)-components of the two vectors.

**Skip dot product**

If we look at the distributive law of multiplication in evaluating our dot product, we discover several important results.

\[ \vec{A} \cdot \vec{B} = (a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \]

\[ = a_x b_x + a_y b_y + a_z b_z \]

\[ = a_x b_x + a_y (b_y + b_z) \]

\[ + a_z (b_x + b_y + b_z) \]

\[ = a_x b_x + a_y b_y + a_z b_z \]

\[ = a_x b_x + a_y b_y + a_z b_z \]

\[ = a_x b_x + a_y b_y + a_z b_z \]
This result tells us that the following relationships must be true:

\[ a_x \hat{x} \cdot b_x = a_x b_x \]
\[ a_y \hat{y} \cdot b_y = a_y b_y \]
\[ a_z \hat{z} \cdot b_z = a_z b_z \]

Or generally...

\[ a_x \hat{x} \cdot b_x = a_x b_x \]
\[ a_y \hat{y} \cdot b_y = a_y b_y \]
\[ a_z \hat{z} \cdot b_z = a_z b_z \]

We can analyze the other two terms (for the \(y\)- and \(z\)-dimensions of \(A\)) and find

\[ \hat{y} \cdot \hat{z} = 0 \]
\[ \hat{z} \cdot \hat{y} = 0 \]
\[ \hat{y} \cdot \hat{y} = 1 \]
\[ \hat{z} \cdot \hat{z} = 1 \]

Finally, the scalar product also obeys the distributive law of multiplication.

\[ \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \]

I leave this as an exercise to you to verify on your own. The proof is rather simple and straightforward.

Scalar products can also be interpreted as the scalar product of length of one vector and the length of the projection of the second vector onto the first.

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

Now, I’ve given you two very different looking expressions for the scalar product of 2 vectors.

\[ \vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z \]

Which one is correct?
(The general proof of this is a subject for linear algebra, but here’s a specific case that illustrates the thinking.)

For any two 2-D vectors, I can choose my coordinate system such that one of the vectors lies entirely along the $x$-axis.

Let’s choose vector $A$ to lie on the $x$-axis.

Proof

Let’s choose vector $A$ to lie on the $x$-axis.

These two definitions of the scalar product provide a powerful tool with which we can determine the angle between any two vectors!

Find the angle between the following two vectors:

$\vec{A} = 2\hat{x} + 3\hat{y}$

$\vec{B} = -1\hat{x} + 2\hat{y}$

Worksheet #1a

Now recall that our definition of work is

$W = F_s \Delta s$

$W = (F \cos \theta) \Delta s$

where $\theta$ is the angle between the force $F$ and the displacement $\Delta s$.
This expression, however, is just that of a scalar product between the force vector and the displacement vector!

\[ W = \mathbf{F} \cdot \mathbf{s} \]

Valid only if the force is constant!

So, a nice, shorthand way to express work is

\[ W = \mathbf{F} \cdot \Delta \mathbf{s} \]

The rate at which work is done.

\[ \mathbf{P} = \frac{\Delta W}{\Delta t} \]

Notice the average power output is simply the slope of the chord! (Look familiar?)

We can also calculate the instantaneous power being delivered by a force.

\[ P(t) = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \mathbf{F} \cdot \Delta \mathbf{s} = \mathbf{F} \cdot \mathbf{v}(t) \]

Notice that Power is also a scalar quantity.

A 600 kg elevator starts from rest and is pulled upward by a motor with a constant acceleration of 2 m/s\(^2\) for 3 seconds. What is the average power output of the motor during this time period?

1) 59,920 W
2) 21,240 W
3) 17,640 W
4) 3,600 W

We spent the first 8 weeks of class examining the motion of point mass objects. We now look more carefully at real, extended objects that move along curved paths and the forces responsible for their motion.
To make our life simpler, we’re going to use the angular unit of measure known as the **radian** when discussing the motion of objects in circular arcs.

Let’s spend a little time motivating our choice of radians over degrees as the unit of choice for measuring angles.

Let’s start with the following question: **What is π?**

We all probably remember the numerical value of π (3.14159…). But from where does this numerical value come?

π is exactly the ratio of the circumference of a circle to its diameter.

True for ANY circle!

Circumference

Diameter

C = πd

What if we plotted the circumferences of a variety of circles against their diameters, we would see...

And if we plotted the circumferences of a variety of circles against their diameters, we would see...

<table>
<thead>
<tr>
<th>Slope = π</th>
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</thead>
<tbody>
<tr>
<td>C = πd</td>
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What if we changed to a plot of circumference versus radius? How would the plot change?

Circumference

Radius

C = 2πr

Slope = 2π

Notice our graph is now twice as steep as it was before.

Okay, this stuff all SEEMS pretty obvious. Just where are we headed?

How do we compute ARC-LENGTH along a circle?

Notice that this property – namely that the value of quantity s/r at a given θ does not depend upon the size of the circle property – is shared with the trigonometric functions (sine, cosine, tangent, etc.).

s = θr

Where θ is measured in radians!

How does the ratio of s/r change as we increase the size of our circle?

No Change!

NOTE: The angle θ is still the same.
Here $\theta$ is measured in radians.

Notice that the ratio $s/r$ varies linearly with the size of the angle $\theta$.

The measure of radians provides a natural system with which to measure angles.

This is in contrast with the system of degrees, which resulted from the completely arbitrary decision to put 360 of them in one complete revolution around a circle.

Now that we’ve seen that the natural system in which to measure in the “angular world” is the radian system, we are free to explore angular motions.

Completely analogous to our discussion of the linear motion quantities of displacement velocity acceleration, we can now define the angular quantities:

- Angular displacement
- Angular velocity
- Angular acceleration

Angular displacement

Positive changes correspond to counter-clockwise rotations.

$$\Delta \theta = \theta_f - \theta_i$$

Angular velocity

And if we know it takes the yellow ball a time $\Delta t$ to move through the angle $\Delta \theta$, then we can calculate an angular velocity...

$$\omega = \frac{\Delta \theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i}$$

Angular velocity is denoted by the symbol $\omega$.

Notice that the angular displacement can be directly translated into a linear distance traveled using our arc length from before...

$$s = r \Delta \theta = r(\theta_f - \theta_i)$$

Just as with its linear counterpart, we can examine a graphical representation of this quantity to better understand its meaning.
Just as in the linear case, where we examined the rate of change of the velocity, in the angular case, we can examine the rate of change of the angular velocity, which we call the angular acceleration.

Angular acceleration is denoted by the symbol $\alpha$.

As we did for the linear quantity and for angular velocity, let's look at the graph.

Again, these are completely analogous to what we derived for the kinetic equations of a linear system with constant linear acceleration!
A wheel rotates with constant angular acceleration of $\alpha_0 = 2 \text{ rad/s}^2$. If the wheel starts from rest, how many revolutions does it make in 10 s?

Worksheet #4

We’ve seen how arc length relates to an angle swept out:

$$s = r\Delta\theta = r(\theta_f - \theta_i)$$

Let’s look at how our angular velocity and acceleration relate to linear quantities.

For an object moving in a circle with a constant linear speed (a constant angular velocity), the instantaneous velocity vectors are always tangent to the circle of motion.

The magnitude of the tangential velocity can be found from our relationship of arc length to angle...

$$s = r\Delta\theta \quad \frac{s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \quad v_{\text{tan}} = r\omega$$

Two wheels, A and B, are rotated with constant angular acceleration of $\alpha = 2 \text{ rad/s}^2$. Both wheels start from rest. If the radius of wheel A is twice the radius of wheel B, how does the angular velocity of wheel A compare to wheel B at time $t = 10$ s?

1) 1/4 as great
2) 1/2 as great
3) Same
4) 2 times as great
5) 4 times as great

Worksheet #5

For an object moving around a circle with a changing angular velocity, and hence a changing tangential velocity, the instantaneous tangential acceleration is NON-ZERO!

Let’s look at how the tangential velocity changes with time in such a case:

$$\frac{\Delta v_{\text{tan}}}{\Delta t} = \frac{v_{\text{tan}}(t_f) - v_{\text{tan}}(t_i)}{t_f - t_i} = \frac{\omega_f - \omega_i}{t_f - t_i}$$

Worksheet #6
For an object moving around a circle with a changing angular velocity, and hence a changing tangential velocity, the **instantaneous tangential acceleration** is NON-ZERO!

Let’s look at how the tangential velocity changes with time in such a case:

\[
a_{\text{tan}} = \frac{r \Delta \omega}{\Delta t} = r \alpha
\]

The tangential acceleration, however, is not the only acceleration we need to consider in problems of circular motion...

The purple arrows represent the direction of the **CENTRIPETAL ACCELERATION**, which always points towards the center of the circle.

Recall our definition of centripetal acceleration:

\[
a_c = \frac{v^2}{r} = (r \omega)^2 = r \omega^2
\]

Notice: Our new form uses angular velocity!