Physics 111

Lecture 17

Thursday, October 28, 2004

- Ch 8: Work done by force at an angle
  Power

- Ch 10: Rotational Kinematics
  angular displacement
  angular velocity
  angular acceleration
Announcements

The Physics 111 Help Session

- Wednesday, 8 - 9 pm in NSC 118/119
- Sunday, 6:30 - 8 pm in CCLIR 468
This week’s lab will be a workshop of physics. Bring your ranking tasks book. **No quiz.**
Worksheet #1

Concept Quiz!

Springs & Energy

CAUTION
WATCH YOUR STEP

Ch 8: Conservation of Energy

Phys 111

Thurs Oct 28
A spring-loaded toy dart gun is used to shoot a dart straight up in the air, and the dart reaches a maximum height of 24 m. The same dart is shot straight up a second time from the same gun, but this time the spring is compressed only half as far before firing. How far up does the dart go this time, neglecting friction and assuming an ideal spring?

1. 96 m  2. 48 m
3. 24 m  4. 28 m
5. 6 m  6. 3 m
7. impossible to determine

PI, Mazur (1997)
Let's now look at a force that is applied at an angle to the direction of motion.

How does our problem change?

Frictionless pond of ice
If the block remains on the ice, and the man keeps the angle between the rope and the ice constant, and the man exerts a constant force, the work done on the block is….
We look at the component of the force that is in the direction of motion of the object (i.e., along the direction of $s$).

Distance = $\Delta s$
Work! 

\[ \mathcal{W} = F_s \Delta s = T_s \Delta s = (T \cos \theta) \Delta s \] 

Frictionless pond of ice
Notice: Work is a scalar quantity, which means that you do NOT specify a direction associated with work. Work only has a magnitude.

Let’s introduce a new mathematical operation to express the type of product we need to calculate in order to correctly compute work.
The scalar (or dot) product of any two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by the expression

\[ \mathbf{A} \cdot \mathbf{B} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \]

where

\[ \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \]

and

\[ \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \]

is given by the expression
Scalar Product (a.k.a. Dot Product)

\[ \vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z \]

Which mathematically says:
- multiply the $x$-components of the two vectors;
- add the result to the product of the $y$-components of the two vectors; and finally
- add the result to the product of the $z$-components of the two vectors.
If we look at the distributive law of multiplication in evaluating our dot product, we discover several important results.

\[ \vec{A} \cdot \vec{B} = (a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \]

\[ = a_x \hat{x} \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \]
\[ + a_y \hat{y} \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \]
\[ + a_z \hat{z} \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \]
This result tells us that the following relationships must be true:

\[ a_x \hat{x} \cdot b_x \hat{x} = a_x b_x \]

\[ a_x \hat{x} \cdot b_y \hat{y} = 0 \]

\[ a_x \hat{x} \cdot b_z \hat{y} = 0 \]

Or generally...

\[ \hat{x} \cdot \hat{x} = 1 \]

\[ \hat{x} \cdot \hat{y} = 0 \]

\[ \hat{x} \cdot \hat{z} = 0 \]
We can analyze the other two terms (for the $y$- and $z$- dimensions of $A$) and find

\[
\hat{y} \cdot \hat{z} = 0
\]
\[
\hat{z} \cdot \hat{x} = 0
\]
\[
\hat{y} \cdot \hat{y} = 1
\]
\[
\hat{z} \cdot \hat{y} = 0
\]
\[
\hat{y} \cdot \hat{z} = 0
\]
\[
\hat{z} \cdot \hat{z} = 1
\]
What is also clear from our definition of this operation is that scalar products are \textit{commutative}.

\[
\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}
\]

\[
\vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z
\]

\[
\vec{B} \cdot \vec{A} = b_x a_x + b_y a_y + b_z a_z
\]
Finally, the scalar product also obeys the distributive law of multiplication.

\[ \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \]

I leave this as an exercise to you to verify on your own. The proof is rather simple and straightforward.
Scalar products can also be interpreted as the scalar *product of length of one vector and the length of the projection of the second vector onto the first.*

\[
\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta
\]
Now, I’ve given you two very different looking expressions for the scalar product of 2 vectors.

\[ \vec{A} \cdot \vec{B} = a_x b_x + a_y b_y + a_z b_z \]

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \]

Which one is correct?
(The general proof of this is a subject for linear algebra, but here's a specific case that illustrates the thinking.) **skip**

For any two 2-D vectors, I can choose my coordinate system such that one of the vectors lies entirely along the $x$-axis.

Let’s choose vector $A$ to lie on the $x$-axis.
Using the 2nd definition of the scalar product

\[ \vec{A} \cdot \vec{B} = A_x | \vec{B} | \cos \theta = A_x \left( \sqrt{B_x^2 + B_y^2} \right) \left( \frac{B_x}{\sqrt{B_x^2 + B_y^2}} \right) \]

\[ \vec{A} \cdot \vec{B} = A_x B_x \]
Using the 1st definition of the scalar product

\[ \vec{A} \cdot \vec{B} = \begin{align*} A_x B_x + A_y B_y \end{align*} \]
These two definitions of the scalar product provide a powerful tool with which we can determine the angle between any two vectors!

Find the angle between the following two vectors:

\[ \vec{A} = 2\hat{x} + 3\hat{y} \quad \vec{B} = -1\hat{x} + 2\hat{y} \]

Worksheet #1a
Find the angle between the following two vectors:

\[ \vec{A} = 2\hat{x} + 3\hat{y} \quad \vec{B} = -1\hat{x} + 2\hat{y} \]

\[ \vec{A} \cdot \vec{B} = (2)(-1) + (3)(2) = 4 \]

\[ |\vec{A}| |\vec{B}| \cos \theta = \left(\sqrt{2^2 + 3^2}\right)\left(\sqrt{(-1)^2 + 2^2}\right) \cos \theta \]

\[ |\vec{A}| |\vec{B}| \cos \theta = \left(\sqrt{65}\right) \cos \theta \quad \cos \theta = \frac{4}{\sqrt{65}} \]

\[ \theta = 60.3^\circ \]
Now recall that our definition of work is

\[ W = F_s \Delta s \]

\[ W = (F \cos \theta) \Delta s \]

where \( \theta \) is the angle between the force \((F)\) and the displacement \((\Delta s)\)
This expression, however, is just that of a scalar product between the force vector and the displacement vector!

So, a nice, shorthand way to express work is

\[ W = (F \cos \theta) \Delta s \]

Valid only if the force is constant!
The rate at which work is done.

\[ \overline{P} = \frac{\Delta W}{\Delta t} \]

Notice the average power output is simply the slope of the chord! (Look familiar?)
We can also calculate the instantaneous power being delivered by a force.

\[ P(t) = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \bar{F} \cdot \Delta \bar{s} \]

\[ = \bar{F} \cdot \bar{v}(t) \]

Notice that *power* is also a *scalar* quantity.
POWER!!!

\[ P = \vec{F} \cdot \vec{v} \]

\[ [P] = [F][v] \]

\[ [P] = \text{Nm} / \text{s} \]

\[ [P] = \text{J} / \text{s} = \text{W} \]
A 600 kg elevator starts from rest and is pulled upward by a motor with a constant acceleration of 2 m/s² for 3 seconds. What is the average power output of the motor during this time period?

1) 59,920 W
2) 21,240 W
3) 17,640 W
4) 3,600 W

Worksheet #2

Ch 7: Work and Kinetic Energy
A 600 kg elevator starts from rest and is pulled upward by a motor with a constant acceleration of 2 m/s² for 3 seconds. What is the average power output of the motor during this time period?

Let’s first figure out the *force* delivered by the motor, which we’ll assume equals the tension...

\[ F_{\text{net}} = ma = (600 \text{ kg})(2 \text{ m/s}^2) \]

\[ F_{\text{net}} = 1200 \text{ N} = F_{\text{motor}} - W = F_{\text{motor}} - mg \]

\[ 1200 \text{ N} = F_{\text{motor}} - (600 \text{ kg})(9.8 \text{ m/s}^2) = F_{\text{motor}} - 5880 \text{ N} \]

\[ F_{\text{motor}} = 5880 \text{ N} + 1200 \text{ N} = 7080 \text{ N} \]
A 600 kg elevator starts from rest and is pulled upward by a motor with a constant acceleration of 2 m/s² for 3 seconds. What is the average power output of the motor during this time period?

Now we need to determine the work done by the motor...

\[ W = F \Delta s \quad \text{But we don’t know } \Delta s, \text{ so…} \]

\[ s = s_0 + v_0 t + 0.5at^2 = 0 + 0 + 0.5(2 \text{ m/s}^2)(3 \text{ s})^2 = 9 \text{ m} \]

\[ W = (7080 \text{ N})(9 \text{ m}) = 63720 \text{ J} \]

\[ \bar{P} = \frac{W}{\Delta t} = \frac{63720 \text{ J}}{3 \text{ s}} = 21240 \text{ W} \]
We spent the first 8 weeks of class examining the motion of *point mass* objects. We now look more carefully at real, extended objects that move along curved paths and the *forces* responsible for their motion.
To make our life simpler, we’re going to use the angular unit of measure known as the **radian** when discussing the motion of objects in circular arcs.

**Why?**

Let’s spend a little time motivating our choice of **radians over degrees** as the unit of choice for measuring angles.
Let’s start with the following question: What is $\pi$?

We all probably remember the numerical value of $\pi$ (3.14159…). But from where does this numerical value come?

$\pi$ is exactly the ratio of the circumference of a circle to its diameter.

True for ANY circle!
And if we plotted the circumferences of a variety of circles against their diameters, we would see...

\[ C = \pi d \]

What if we changed to a plot of circumference versus radius? How would the plot change?
Circumference \( C \) vs Radius \( r \)

Ch 10: Rotational Kinematics

\[ C = 2\pi r \]

Slope = \( 2\pi \)

Notice our graph is now twice as steep as it was before.

Okay, this stuff all SEEMS pretty obvious. Just where are we headed?

How do we compute ARC-LENGTH along a circle?
How does the ratio of $s/r$ change as we increase the size of our circle?

NOTE: The angle $\theta$ is still the same.
Notice that this property – namely that the value of quantity \( s/r \) at a given \( \theta \) **does not depend upon the size of the circle** property – is shared with the trigonometric functions (sine, cosine, tangent, etc.).

\[
\cos \theta = \frac{x_1}{r_1} = \frac{x_2}{r_2}
\]
Here $\theta$ is measured in radians.

Notice that the ratio $s/r$ varies linearly with the size of the angle $\theta$.

The measure of radians provides a natural system with which to measure angles.

This is in contrast with the system of degrees, which resulted from the completely arbitrary decision to put 360 of them in one complete revolution around a circle.
Now that we’ve seen that the natural system in which to measure in the “angular world” is the radian system, we are free to explore angular motions.

Completely analogous to our discussion of the linear motion quantities of displacement, velocity, acceleration, we can now define the angular quantities of angular displacement, angular velocity, angular acceleration.
Angular Displacement

Positive changes correspond to counter-clockwise rotations.

\[ \Delta \theta = \theta_f - \theta_i \]
Notice that the angular displacement can be directly translated into a linear distance traveled using our arc length from before...

\[ s = r \Delta \theta = r(\theta_f - \theta_i) \]
And if we know it takes the yellow ball a time $\Delta t$ to move through the angle $\Delta \theta$, then we can calculate an angular velocity...

Angular velocity is denoted by the symbol $\omega$.

$\omega = \frac{\Delta \theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i}$
Angular Velocity

Just as with its linear counterpart, we can examine a graphical representation of this quantity to better understand its meaning.

\[
\text{slope} = \omega \\
\text{Instantaneous angular velocity}
\]

\[
\text{slope} = \bar{\omega} = \frac{\Delta \theta}{\Delta t} \\
\text{Average angular velocity}
\]
Angular Velocity

\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t} \]

[\[ \bar{\omega} \] = \frac{[\Delta \theta]}{[\Delta t]}]

[\[ \bar{\omega} \] = \frac{\text{rad}}{s} = s^{-1}]
Worksheet #3

Concept Quiz!

Angular Velocity

Ch 10: Rotational Kinematics

Thurs Oct 28

Phys 111
A ladybug sits at the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug’s angular speed is

1. half the ladybug’s.
2. the same as the ladybug’s.
3. twice the ladybug’s.
4. impossible to determine

Pl, Mazur (1997)
Just as in the linear case, where we examined the rate of change of the velocity, in the angular case, we can examine the rate of change of the angular velocity, which we call the angular acceleration.

Angular acceleration is denoted by the symbol $\alpha$. 

$$\vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t} = \frac{\vec{\omega}_f - \vec{\omega}_i}{t_f - t_i}$$
As we did for the linear quantity and for angular velocity, let’s look at the graph.

**Angular Acceleration**

- **Instantaneous angular acceleration**
  \[ \text{slope} = \ddot{\alpha} \]

- **Average angular acceleration**
  \[ \text{slope} = \ddot{\alpha} = \frac{\Delta \ddot{\omega}}{\Delta t} \]
Angular Acceleration

\[ \vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t} \]

[\vec{\alpha}] = \left[ \frac{\Delta \vec{\omega}}{\Delta t} \right]

\[ [\vec{\alpha}] = \frac{\text{rad/s}}{\text{s}} = \text{s}^{-2} \]
Equations for Systems Involving Rotational Motion with Constant Angular Acceleration

Again, these are completely analogous to what we derived for the kinetic equations of a linear system with constant linear acceleration!

\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \vec{\omega} = \vec{\omega}_0 + \vec{\alpha} t \]
A wheel rotates with constant angular acceleration of \( \alpha_0 = 2 \text{ rad/s}^2 \). If the wheel starts from rest, how many revolutions does it make in 10 s?

**Worksheet #4**

\[
\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2
\]

\[
\Delta \theta = \omega_0 t + \frac{1}{2} \alpha t^2 = 0 + \frac{1}{2} \left(2 \frac{\text{rad}}{s^2}\right)(10 \text{ s})^2
\]

\[
\Delta \theta = 100 \text{ rad}
\]

\[
1 \text{ rev} = 2\pi \text{ rad}
\]

\[
\Delta \theta = (100 \text{ rad}) \frac{1 \text{ rev}}{2\pi \text{ rad}} = \frac{50}{\pi} \text{ rev} = 15.9 \text{ rev}
\]
We’ve seen how arc length relates to an angle swept out:

\[ s = r \Delta \theta = r(\theta_f - \theta_i) \]

Let’s look at how our **angular velocity** and **acceleration** relate to linear quantities.
For an object moving in a circle with a constant linear speed (a constant angular velocity), the instantaneous velocity vectors are always tangent to the circle of motion.

The magnitude of the tangential velocity can be found from our relationship of arc length to angle...

\[ s = r \Delta \theta \]

\[ \frac{s}{\Delta t} = r \frac{\Delta \theta}{\Delta t} \]

\[ v_{\text{tan}} = r \omega \]
Two wheels, A and B, are rotated with constant angular acceleration of $\alpha = 2 \text{ rad/s}^2$. Both wheels start from rest. If the radius of wheel A is twice the radius of wheel B, how does the angular velocity of wheel A compare to wheel B at time $t = 10 \text{ s}$?

1) 1/4 as great
2) 1/2 as great
3) Same
4) 2 times as great
5) 4 times as great
Two wheels, A and B, are rotated with constant angular acceleration of $\alpha = 2 \text{ rad/s}^2$. Both wheels start from rest. If the radius of wheel A is twice the radius of wheel B, how does the tangential velocity of wheel A compare to wheel B at time $t = 10 \text{ s}$?

1) 1/4 as great
2) 1/2 as great
3) Same
4) 2 times as great
5) 4 times as great

Worksheet #6
For an object moving around a circle with a changing angular velocity, and hence a changing tangential velocity, the instantaneous tangential acceleration is NON-ZERO!

Let’s look at how the tangential velocity changes with time in such a case:

\[
\Delta v_{\tan} = \frac{v_{\tan f} - v_{\tan i}}{t_f - t_i} = \frac{r\omega_f - r\omega_i}{t_f - t_i}
\]
Ch 10: Rotational Kinematics

For an object moving around a circle with a changing angular velocity, and hence a changing tangential velocity, the instantaneous tangential acceleration is NON-ZERO!

Let’s look at how the tangential velocity changes with time in such a case:

\[ a_{\text{tan}} = \frac{r\Delta \omega}{\Delta t} = r\alpha \]
The tangential acceleration, however, is not the only acceleration we need to consider in problems of circular motion...

The purple arrows represent the direction of the **CENTRIPETAL ACCELERATION**, which always points towards the center of the circle.
Recall our definition of centripetal acceleration:

$$a_c = \frac{v_r^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

Notice: Our new form uses angular velocity!